

# Extended Lorenz Models and Time Dependent First Integrals

W.-H. Steeb

Rand Afrikaans University, Department of Applied Mathematics and Nonlinear Studies,  
Auckland Park 2006, South Africa  
email: WHS@RAU3.RAU.AC.ZA, P.O. Box 524

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We show how computer algebra can be used to find time dependent first integrals of dissipative dynamical systems. Extended Lorenz models are studied as examples.

The Lorenz model [1] and extended versions thereof are autonomous systems of first-order differential equations

$$\frac{du}{dt} = V(u), \quad (1)$$

where  $u = (u_1, u_2, \dots, u_n)^T$ . The functions  $V_j$  are polynomials. Obviously, they also depend on bifurcation parameters. Furthermore we have  $\text{div } V < 0$ . We call such a system dissipative. Such systems can admit time dependent first integrals for certain values of the bifurcation parameters in system (1). A special ansatz for such a first integral was introduced by Steeb [2] (see also Kus [3], Schwarz and Steeb [4], Kowalski and Steeb [5], Almeida and Moreira [6], Steeb [7]). The ansatz for the time dependent first integral is given by

$$I(u(t), t) := f(u(t)) \exp(\varepsilon t), \quad (2)$$

where  $f$  is a polynomial

$$f(u) = a_0 + \sum_{k=1}^n a_k u_k + \sum_{k_1=1}^n \sum_{k_2=1}^{k_1} b_{k_1 k_2} u_{k_1} u_{k_2} + \sum_{k_1=1}^n \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} c_{k_1 k_2 k_3} u_{k_1} u_{k_2} u_{k_3} + \dots \quad (3)$$

The coefficients  $a_k, b_{k_1 k_2}, c_{k_1 k_2 k_3}, \dots$  and the real number  $\varepsilon$  are determined from the condition that  $I$  is a first integral, i.e.,

$$\frac{d}{dt} I(u(t), t) = 0. \quad (4)$$

Condition (4) leads to

$$\sum_{j=1}^n \frac{\partial f}{\partial u_j} V_j + \varepsilon f = 0. \quad (5)$$

Reprint requests to Prof. Dr. W.-H. Steeb, Department of Applied Mathematics, Rand Afrikaans University, Post Office Box 524, Auckland Park 2006/South Africa.

The left hand side of (5) is a polynomial  $P(u)$ , which can be written as

$$P(u) = \sum_{k_1=0} \dots \sum_{k_n=0} p_{k_1 k_2 \dots k_n} u_1^{k_1} u_2^{k_2} \dots u_n^{k_n}. \quad (6)$$

Thus from (5) we find that

$$p_{k_1 k_2 \dots k_n} = 0, \quad k_1 = 0, 1, \dots, \dots, k_n = 0, 1, \dots \quad (7)$$

We apply computer algebra to evaluate (5) and (6) and then (7). Then we solve the determining equations given by (7). In the program we first have to generate the polynomial  $f$  given by (3). Here we have to take into account that  $u_i u_k = u_k u_i$ , etc. In the second step we evaluate condition (5). Next we have to separate out the different coefficients of the polynomial. This leads to a nonlinear system of algebraic equations owing to the parameter  $\varepsilon$ . The nonlinearity comes into play because we include the exponential  $\exp(\varepsilon t)$  into ansatz (2). We find terms of the form  $a_j \varepsilon, b_{jk} \varepsilon$ . Moreover the number of equations is larger than the number of unknowns.

We consider two extended Lorenz models. The first one with dimension  $n=5$  and the second one with  $n=4$ . For the second one we give the REDUCE program, which can be easily adapted to the first case.

Consider first the following extended Lorenz model [8]:

$$\begin{aligned} \frac{du_1}{dt} &= \sigma(u_3 - u_1) + \sigma S u_5, \\ \frac{du_2}{dt} &= -b u_2 + u_1 u_3, \\ \frac{du_3}{dt} &= -u_1 u_2 + r u_1 - b u_3, \\ \frac{du_4}{dt} &= u_1 u_5 - b \tau u_4 - b \tau u_2, \\ \frac{du_5}{dt} &= -u_1 u_4 + r u_1 - \tau u_5 + \tau u_3, \end{aligned} \quad (8)$$

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where  $\sigma, r, b, S, \tau \in \mathbb{R}^+$ . Notice that  $\text{div } V = -\sigma - b - 1 - b\tau - \tau$ . In our evaluation we include polynomials up to degree 2. We find the following time dependent first integrals of the form (2): If  $2\sigma + b\tau$  and  $\tau(1-S) = 0$ , then

$$I(u, t) = (2\sigma u_2 + 2S\sigma u_4 - u_1^2) \exp(2\sigma t) \quad (9)$$

is a time dependent first integral. If  $\sigma = 1, b = 1, \tau = 2$ , then

$$I(u, t) = (2Sr u_4 + 4Sr u_2 - r u_1^2 - (2S - 1)(u_2^2 + u_3^2)) \cdot \exp(2t) \quad (10)$$

is a time dependent first integral. If some of the bifurcation parameters are equal to zero, then we find additional first integrals. For example, if  $\tau = r = 0, b = 1, \varepsilon = 1$ , then  $(u_3 u_5 + u_2 u_4) \exp(t)$  is a first integral.

Consider now another extended Lorenz model [9]:

$$\begin{aligned} \frac{du_1}{dt} &= -\sigma u_1 + \sigma u_2, \\ \frac{du_2}{dt} &= r u_1 - u_2 - u_1 u_3 + \varepsilon u_4 \left(1 + \frac{u_3}{r}\right), \\ \frac{du_3}{dt} &= -b u_3 + u_1 u_2, \\ \frac{du_4}{dt} &= -u_4 - \varepsilon u_2 \left(1 + \frac{u_3}{r}\right), \end{aligned} \quad (11)$$

where  $\sigma, b, r, \varepsilon \in \mathbb{R}^+$ . Notice that  $\text{div } V = -\sigma - b - 2$ . It shows chaotic behaviour for a wide range of its parameters. However, for various values of the parameters there are time dependent first integrals of the form (2). Here we find, up to degree two in the polynomial  $f$ , the first integrals: If  $\varepsilon = b = 2\sigma$ , then

$$I(u, t) = (-2\sigma u_3 + u_1^2) \exp(2\sigma t) \quad (12)$$

is a first integral. If  $\varepsilon = 2, \sigma = b = 1, r = 2$ , then

$$I(u, t) = (-2u_1^2 + u_2^2 + u_3^2 + u_4^2) \exp(2t) \quad (13)$$

is a first integral.

Next we give the REDUCE program (Steeb and Lewien [10], Steeb [11], Steeb [12], Hearn [13]) which finds the determining equations (5) for the extended Lorenz model (11). We set  $\varepsilon = \text{vep}$ ,  $\varepsilon = \text{ep}$  and  $\sigma = \text{sig}$  in the program.

```
% fint8.red;
operator A, B, U, V, P;
n := 4;
F1 := for k:= 1:n sum A(k)*U(k);
```

```
F2 := for k1:= 1:n sum (for k2:= 1:k1 sum
B(k2, k1)*U(k2)*U(k1));
```

```
% extended Lorenz model equation (11)
```

```
V(1) := -sig*U(1) + sig*U(2);
V(2) := -U(1)*U(3) + r*U(1) - U(2)
+ vep*U(4)*(1 + U(3)/r);
V(3) := U(1)*U(2) - b*U(3);
V(4) := -U(4) - ep*U(2)*(1 + U(3)/r);
```

```
F := A(0) + F1 + F2;
```

```
% condition (5);
```

```
res := (for k:= 1:n sum df(F, U(k))*V(k)) + F*EP;
```

```
% separating out the different terms of the polynomial;
```

```
for j1:= 0:3 do for j2:= 0:3 do for j3:= 0:3 do
for j4:= 0:3 do P(j1, j2, j3, j4) :=
coeffn(coeffn(coeffn(coeffn(res, U(1), j1), U(2), j2), U(3), j3),
U(4), j4);
for j1:= 0:3 do for j2:= 0:3 do for j3:= 0:3 do
for j4:= 0:3 do write "P(", j1, ", ", j2, ", ", j3, ", ", j4, ")
=", P(j1, j2, j3, j4);
```

The output is given by (we only give the nonzero coefficients)

```
P(0, 0, 0, 0) = a(0)*ep$
P(0, 0, 0, 1) = a(4)*ep - a(4) + a(2)*vep$
P(0, 0, 0, 2) = b(4, 4)*ep - 2*b(4, 4) + b(2, 4)*vep$
P(0, 0, 1, 0) = a(3)*(-b + ep)$
P(0, 0, 1, 1) = (a(2)*vep - b(3, 4)*b*r + b(3, 4)*r*ep
- b(3, 4)*r + b(2, 3)*r*vep)/r$
P(0, 0, 1, 2) = (b(2, 4)*vep)/r$
P(0, 0, 2, 0) = b(3, 3)*(-2*b + ep)$
P(0, 0, 2, 1) = (b(2, 3)*vep)/r$
P(0, 1, 0, 0) = -a(4)*ep + a(2)*ep - a(2) + a(1)*sig$
P(0, 1, 0, 1) = -2*b(4, 4)*ep + b(2, 4)*ep - 2*b(2, 4)
+ 2*b(2, 2)*vep + b(1, 4)*sig$
P(0, 1, 1, 0) = (-a(4)*ep - b(3, 4)*r*ep - b(2, 3)*b*r
+ b(2, 3)*r*ep - b(2, 3)*r + b(1, 3)*r*sig)/r$
P(0, 1, 1, 1) = (2*(-b(4, 4)*ep + b(2, 2)*vep))/r$
P(0, 1, 2, 0) = (-b(3, 4)*ep)/r$
P(0, 2, 0, 0) = -b(2, 4)*ep + b(2, 2)*ep - 2*b(2, 2)
+ b(1, 2)*sig$
P(0, 2, 1, 0) = (-b(2, 4)*ep)/r$
P(1, 0, 0, 0) = a(2)*r + a(1)*ep - a(1)*sig$
P(1, 0, 0, 1) = b(2, 4)*r + b(1, 4)*ep - b(1, 4)*sig - b(1, 4)
+ b(1, 2)*vep$
P(1, 0, 1, 0) = -a(2) + b(2, 3)*r - b(1, 3)*b + b(1, 3)*ep
- b(1, 3)*sig$
P(1, 0, 1, 1) = (-b(2, 4)*r + b(1, 2)*vep)/r$
P(1, 0, 2, 0) = -b(2, 3)$
P(1, 1, 0, 0) = a(3) + 2*b(2, 2)*r - b(1, 4)*ep + b(1, 2)*ep
- b(1, 2)*sig - b(1, 2) + 2*b(1, 1)*sig$
P(1, 1, 0, 1) = b(3, 4)$
P(1, 1, 1, 0) = (2*b(3, 3)*r - 2*b(2, 2)*r - b(1, 4)*ep)/r$
P(1, 2, 0, 0) = b(2, 3)$
P(2, 0, 0, 0) = b(1, 2)*r + b(1, 1)*ep - 2*b(1, 1)*sig$
P(2, 0, 1, 0) = -b(1, 2)$
P(2, 1, 0, 0) = b(1, 3)$
```

We obviously assume that  $\varepsilon \neq 0$ . We also assume that all the bifurcation parameters are nonzero. The 21 unknowns are  $a(0), a(1), \dots, b(4, 4), \varepsilon$ . The number of equations is 27. On inspecting the output we find that

$$b_{13} = b_{12} = b_{23} = b_{24} = b_{34} = 0. \quad (14)$$

It follows that  $a_0 = a_1 = a_2 = a_4 = 0$ . Therefore our system simplifies to

$$\begin{aligned} b_{44}(\varepsilon - 2) &= 0, \quad a_3(-b + \varepsilon) = 0, \quad b_{33}(-2b + \varepsilon) = 0, \\ -2b_{44}\varepsilon + 2b_{22}\varepsilon + b_{44}\sigma &= 0, \quad -b_{44}\varepsilon + b_{22}\varepsilon = 0, \\ b_{22}(\varepsilon - 2) &= 0, \quad b_{14}(\varepsilon - \sigma - 1) = 0, \\ a_3 + 2rb_{22} - b_{14}\varepsilon + 2b_{11}\sigma &= 0, \\ 2rb_{33} - 2rb_{22} - b_{14}\varepsilon &= 0, \quad b_{11}(\varepsilon - 2\sigma) = 0. \end{aligned} \quad (15)$$

There are two strategies to solve this system. With the first one we try to solve the simplest equation first and then insert its solution into the remaining equations. Then the next simplest equation is solved and inserted into the remaining equations and so on. With the second strategy we realize on inspection of system (15)

that  $\varepsilon$  should satisfy one of the following relations:  $\varepsilon = 1$  or  $\varepsilon = 2$  or  $\varepsilon = b$  or  $\varepsilon = 2b$  or  $\varepsilon = \sigma$  or  $\varepsilon = 2\sigma$  or  $\varepsilon = \sigma + 1$ . With one of these assumptions system (15) becomes linear. Let us first consider the case  $\varepsilon = 2\sigma$ . Then  $b_{11}$  is arbitrary and we arrive at the first integral (12). Now let  $\varepsilon = 2$ . Then  $b_{22}$  or  $b_{44}$  is arbitrary and we find the first integral (13).

If we include higher order polynomials we also find first integrals of the form (2) with the polynomials of degree four. Here we must take into account that if  $I_1$  and  $I_2$  are first integrals, then  $I_1 I_2$  is also a first integral.

The described approach can also be applied to conservative Hamiltonian systems. Here the time dependence would involve  $\sin(\omega_1 t)$  and  $\cos(\omega_2 t)$  instead of  $\exp(\varepsilon t)$  for dissipative systems.

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